Regime Behaviour and Predictability Properties of the Atmospheric Circulation Studied with Limited-Resolution Spectral Models

H.E. de Swart
Institute of Meteorology and Oceanography, State University of Utrecht
Princetonplein 5, 3584 CC Utrecht, The Netherlands

A brief review is presented of a project carried out at the CWI during the years 1983-1987. These investigations were supported by the Netherlands Foundation for the Technical Sciences (STW), future Technical Science Branch of the Netherlands Organization for Scientific Research (NWO). The aim was to obtain a better understanding of the chaotic properties of the atmosphere, which may contribute to the development of long-range weather prediction models. It is argued that a method for investigating this problem is to analyse highly simplified atmospheric spectral models, since the results may provide clues on how to analyse more complicated models as well as real data. It appears that low-order models possess multiple equilibria, with the corresponding flow patterns resembling large-scale preferent states of the atmospheric circulation. Vacillatory behaviour, in which the system alternately visits different flow regimes, is obtained either by adding stochastic perturbations to the equations or by including a sufficient number of modes in the spectral expansions. The predictability properties of these systems are discussed and particular attention is given to the forcing terms which are added to the spectral equations in order to account for the effect of the neglected modes and physical processes not included in the model.

1. Introduction

During 1983-1987 research was done at the CWI in the STW project 'Mathematical methods for the analysis of atmospheric spectral models'. The aim of this study was to obtain a better understanding of the dynamics of the atmospheric circulation in the midlatitudes (roughly between the 30 and 60 degree latitude), especially in relation to the problem of long-range weather predictions. Modern weather forecasts are based on the results of detailed and complicated numerical models, such as that of the European Centre for Medium Range Weather Forecasting (ECMWF) in Reading, England. It has long been assumed that the period over which the weather is predicted could be increased forever if the numerical models would be further improved and the initial state (determined by means of observations) better prescribed. However, nowadays it is known that there is a fundamental limit to the period over which the weather can be predicted, i.e., it cannot be enlarged by carrying out more and better observations. In most cases this predictability horizon of the atmosphere is encountered after a period of 5 to 14 days (OPSTEEGH [21]). As a consequence, it is not possible to give forecasts for periods longer than about two weeks. Unfortunately, there is precisely a strong need for accurate long-range weather predictions. This is because on time scales between a week and three months frequently climatic anomalies occur which have large social consequences, for example excessive droughts, heat-waves, etc.

In order to understand why forecasts fail on the long term we first present some qualitative arguments. The weather as we (in midlatitudes) experience it is the result of day to day variations in the geographical distribution of high- and low-pressure belts. These so-called synoptic-scale eddies have typical horizontal dimensions of 1000 km, a life span of about a week and are embedded in a belt of predominantly westerly winds. The latter result from an approximate balance between thermal forcing, due to the equator-pole temperature gradient, and the Coriolis force (induced by the rotation of the earth) acting on a moving fluid. Furthermore, due to the presence of a large-scale topography (to which in particular the Himalaya, Rocky Mountains and the oceans contribute) and thermal differences between land and ocean, ultra-long quasi-stationary waves are generated which give the flow a meandering structure. These planetary waves have much larger dimensions (about 10000 km) and longer lives (of the order of several months) than the synoptic-scale eddies. Thus, the atmospheric circulation is characterized by two distinct scales of motion: a planetary scale and a synoptic scale. Little is known about the subtle interplay between these scales of motion. It appears that synoptic-scale eddies develop spontaneously as initially small perturbations of the locally unstable planetary-scale circulation. Moreover, the planetary-scale flow tends to steer and organize the eddies along preferent paths, which are the stormtracks. On the other hand the eddies themselves influence the evolution of the planetary waves. The consequences for the predictability of the atmospheric circulation were systematically studied by LORENZ [18]. He demonstrated that interactions between different scales of motion are the principal cause for the limited predictability of the atmosphere.

As a result of the feedback between the planetary waves and the synoptic-scale eddies quasi-stable flow configurations occur which cause short-range climatic anomalies. The existence of such large-scale preferent states of the atmospheric circulation (sometimes called weather regimes) has been known for a long time. They can be divided into three major types: zonal (high-index) states with strong western winds and small wave amplitudes, meridional (low-index) states with large waves embedded in a weak zonal flow and transitional states which have characteristics of both the high- and low-index states. Typical flow configurations of these regimes in the European region are shown in Figure 1. The situation in Figure 1c is that of a persistent anticyclone near Scandinavia which blocks the standard passage of depressions over Western Europe, in this way causing persistent weather conditions in this region.

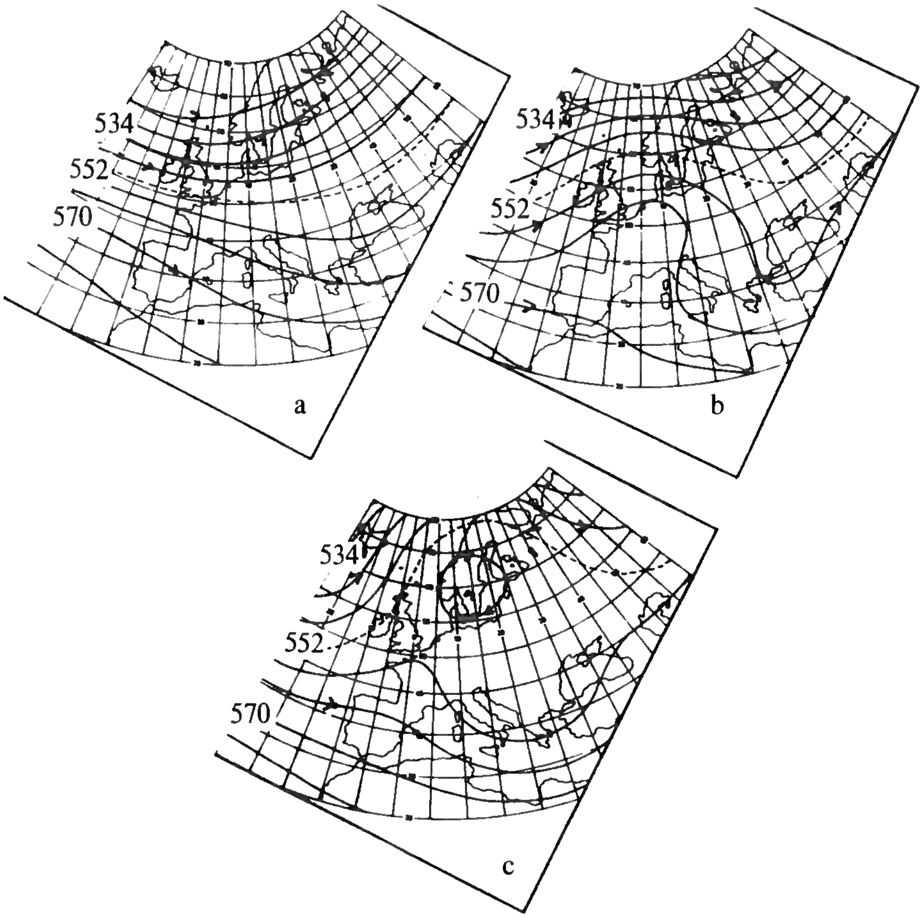

FIGURE 1. Geographical distribution of the height (in geopotential decameter) of the 500 mb level for the Wz (a), HM (b) and HFa (c) winter Grosswetterlagen, which are of zonal, mixed and meridional type, respectively. The isohyps are approximate streamlines of the flow; arrows indicate flow direction. From VAN DIJK ET AL. [29].

Obviously, the atmosphere can be considered as a chaotic system which shows vacillatory behaviour, i.e., it irregularly visits different preferent states. As discussed in DOLE [9] and REINHOLD [23], quasi-stable flow patterns suddenly develop and disappear without any clear indication why. Furthermore, the life span of the weather regimes is highly variable without having a preferent time scale. Within the framework of long-term weather predictions it is important to obtain a better understanding of the dynamics responsible for this vacillatory behaviour. A method for studying this problem consists of analysing highly simplified models which represent qualitative features of the atmospheric circulation. The motivation for doing this is that from the results indications may be found how to consider more complicated models as well as real data. In this way we hope to enhance our understanding in the atmospheric dynamics. This method has been adapted in the STW project mentioned previously and a review of the results will be presented in this paper.

2. Quasi-geostrophic dynamics

In order to study the variability of the atmospheric circulation we should start from the full equations of motion. However, they are too complicated to deal with analytically and therefore they are simplified by the application of scale analysis, being a standard technique in geophysical fluid dynamics, see PEDLOSKY [22]. The method requires an a priori specification of the type of motion to be studied. Next it yields, by means of physical arguments, characteristic scales for the flow with which the equations of motion are written in a dimensionless form. The resulting system will contain several dimensionless parameters. The aim of the method is to find small parameters. Then, by means of standard perturbation techniques, simplified equations are derived which describe the type of motion under consideration.

Here we consider a flow near some central latitude $\phi=\phi_0$ on the Northern Hemisphere distant from equator and pole. Let it have a horizontal (parallel to the earth's surface) length scale k^{-1}, a vertical length scale H (which is the depth of the fluid) and a time scale σ^{-1}, such that

$$H << k^{-1} << r_0, \quad \sigma << f_0 \equiv 2\Omega \sin \phi_0, \tag{2.1}$$

where f_0 is the Coriolis parameter at $\phi=\phi_0$, Ω the angular speed of rotation of the earth and r_0 the radius of the earth. The first condition implies that the flow is nearly horizontal and 2-dimensional. The latter means that to a first approximation the momentum equations reduce to a balance between the Coriolis force and pressure gradient force, which is the geostrophic balance. Clearly, (2.1) is satisfied for large-scale atmospheric motions near $\phi_0=45°$ N where $k^{-1}\sim 10^6$ m, $H\sim 10^4$ m, $\sigma^{-1}\sim 10^5$ s, $f_0=10^{-4}$ s^{-1} and $r_0\sim 6.4\cdot 10^6$ m. Under these conditions it is shown by PEDLOSKY [22] that the equations of motion reduce to one nonlinear partial differential equation. With the additional assumption that the flow is barotropic (i.e., density is a function of pressure only) the result reads (in a dimensionless form)

$$\underset{(1)}{\frac{\partial}{\partial t}\nabla^2\psi} + \underset{(2)}{J(\psi, \nabla^2\psi)} + \underset{(3)}{\gamma J(\psi, h)} + \underset{(4)}{\beta\frac{\partial \psi}{\partial x}} + \underset{(5)}{C\nabla^2(\psi-\psi^*)} = 0. \tag{2.2}$$

This is the barotropic vorticity equation. Here t is time and $\psi(x,y,t)$ a streamfunction to which all state variables (velocities, density, pressure and temperature) are related. At a fixed time the flow is along the streamlines ψ = constant. Furthermore

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}), \quad J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}, \tag{2.3}$$

$$dx = kr_0 \cos\phi_0 d\lambda, \quad dy = kr_0 d\phi,$$

where λ is longitude and ϕ is latitude. In Eq. (2.2) term (1) represents the local change of the relative vorticity $\nabla^2\psi$ (which is the vertical component of the curl of the velocity vector), (2) the advection of vorticity by the flow itself, (3) the production of vorticity due to the presence of a large-scale topography $h(x,y)$ (in particular the high mountains and the oceans) with characteristic amplitude h_0, (4) the planetary vorticity advection due to the variation of the Coriolis parameter with latitude and (5) represents both a damping of vorticity and an external vorticity forcing (modelling the equator-pole temperature gradient) indicated by the function $\psi^*(x,y)$. The dimensionless parameters are

$$\gamma = \frac{f_0 h_0}{\sigma H}, \quad \beta = \frac{\beta_0}{\sigma k} = \frac{2\Omega\cos\phi_0}{\sigma k r_0}, \quad C = \frac{f_0 \delta_E}{2\sigma H}, \tag{2.4}$$

where δ_E is the depth of the boundary layer near the earth's surface in which frictional effects are important. The flow described by Eq. (2.2) is called quasi-geostrophic because the small departures from the geostrophic balance determine the evolution of the flow (PEDLOSKY [22]).

3. DERIVATION OF SPECTRAL MODELS BY GALERKIN PROJECTION TECHNIQUES

The barotropic vorticity equation (2.2) is still difficult to handle, mainly because of its nonlinear structure. A way to obtain approximate solutions is to apply Galerkin projection techniques where explicit use is made of the boundary conditions to the equation (VOIGT ET AL. [30]). This spectral method will be discussed for a specific example. Its application to models used for numerical weather prediction is described by JARRAUD and BAEDE [15]. Consider Eq. (2.2) in a rectangular channel of length L and width $B=(bL/2)$. The dimensionless length and width are 2π and πb, respectively. We investigate the existence of travelling wave solutions in the zonal x-direction. At the boundaries $y=0$ and $y=\pi b$ the meridional velocity component is assumed to be zero and it follows that the mean zonal velocity component over these boundaries should be constant. Consequently, the boundary conditions read

$$\psi(x+2\pi, y, t) = \psi(x,y,t), \tag{3.1}$$

$$\frac{\partial\psi}{\partial x} = 0 \text{ and } \frac{\partial}{\partial t}\int_0^{2\pi}\frac{\partial\psi}{\partial y}dx = 0 \text{ at } y=0, y=\pi b.$$

Applying the spectral method, we expand the streamfunction $\psi(x,y,t)$ in a series of eigenfunctions $\{\phi_j\}_j$ of the Laplace operator ∇^2 with corresponding eigenvalues λ_j, thus

$$\psi(x,y,t) = \sum_j \psi_j(t)\phi_j(x,y), \quad j=(j_1,j_2). \tag{3.2}$$

Each mode $\psi_j\phi_j$ satisfies the boundary conditions and the eigenfunctions are orthonormalized with respect to the domain average. In this case

$$\{\phi_j\} = \sqrt{2}\cos(j_2 y/b) \tag{3.3a}$$

$$\{\phi_j\} = \sqrt{2}\exp(ij_1 x)\sin(j_2 y/b) \tag{3.3b}$$

$$\lambda_j = j_1^2 + \frac{j_2^2}{b^2},\ |j_1|, j_2 = 1,2,....... \tag{3.3c}$$

The functions (3.3a) describe $(0,j_2)$ zonal flow modes (because they are independent of x) and the functions in (3.3b) describe $(|j_1|,j_2)$ wave modes. The topography and forcing streamfunction are represented by

$$h(x,y) = \cos(x)\sin(y/b), \tag{3.4}$$

$$\psi^*(x,y) = \sqrt{2}\{\psi_{01}^*\cos(y/b)+\psi_{02}^*\cos(2y/b)\}.$$

Projecting Eq. (2.2) on the eigenfunctions (3.3), which is called a Galerkin projection, we obtain the spectral model

$$\lambda_j\dot{\psi}_j = \frac{1}{2}\sum_l\sum_m c_{jlm}(\lambda_l-\lambda_m)\psi_l\psi_m + \gamma\sum_l\sum_m c_{jlm}\psi_l h_m + \\ + \sum_l b_{jl}\psi_l - c\lambda_j(\psi_j-\psi_j^*), \tag{3.5}$$

consisting of an infinite number of coupled ordinary differential equations. Here a dot denotes differentiation with respect to time,

$$c_{jlm} = <\phi_j, J(\phi_l,\phi_m)>,\ b_{jl} = \beta<\phi_j, \frac{\partial\phi_l}{\partial x}> \tag{3.6}$$

are the interaction coefficients and $<,>$ denotes an inner product on the domain considered. It appears that nonlinear contributions always occur as triads in which two modes interact and affect the evolution of a third mode. Developing (3.6) using (3.3) we find that there are two types of nonlinear triads: one involving a zonal flow mode and two wave modes and one involving three wave modes. The underlying physical mechanism is discussed in PEDLOSKY [22].

4. THE TRUNCATION PROBLEM

In practice the expansion (3.2) is truncated after a finite number of eigenfunctions. Only the large-scale modes are resolved since it is observed that most energy of quasi-geostrophic flow is contained in the long waves. The result is a dynamical system of the type

$$\dot{x} = f_\mu(x)+F(t) \text{ in } \mathbb{R}^N. \tag{4.1}$$

Here N is the truncation number, $\mathbb{R}^N$ the phase space, $x=(x_1,x_2,...,x_N)$ real-valued velocity amplitudes of the modes (to be specified in the next sections) and $f_\mu(x)$ an N-dimensional vector field depending on x and parameters $\mu=(\mu_1,\mu_2,...,\mu_m)$. Finally, the $F(t)$ represent the effect of the neglected modes on the dynamics of the retained modes. We remark that if (4.1) is used as a forecast model $F(t)$ should also account for the effect of physical processes and boundary conditions not (correctly) incorporated in the model. These forcing terms are unknowns by definition.

A convenient approach in theoretical studies concerning (4.1) is to neglect the effect of the forcing terms a priori and consider the properties of spectral models with increasing truncation numbers. The underlying motivation is that they will at least represent properties of the original vorticity equation. Some formal indications that this idea is correct have been found by CONSTANTIN ET AL. [4]. They showed that for spectral models of the Navier-Stokes equations a minimum number N_s of eigenfunctions could be selected such that solutions of truncated spectral models with $N \geqslant N_s$ and $F(t)=0$ have equal attractor properties as the solutions of the original system. Although it is not clear whether these results are applicable to the barotropic vorticity equation, they at least suggest that it is useful to consider truncated spectral models.

In principle we would like to investigate the properties of (4.1) for arbitrary values of N. However, we remark that it is not possible to carry out such an analysis systematically since the systems have a complicated dynamics due to the large number of nonlinear terms in the equations. Therefore, as a first step, it becomes worthwhile to study low-order spectral models, in which only a few modes are retained, and investigate in what sense they reflect features like transitions between weather regimes and a flow with a limited predictability. An important advantage is that they can be analysed with techniques originating from the theory of dynamical systems, see GUCKENHEIMER and HOLMES [14] and THOMPSON and STEWART [28], whereas from the results indications may be found how to study more complicated models as well as real data.

The structure of the vector fields of the spectral models discussed in this paper is such that nontransient solutions are found in bounded subsets of the phase space. These can be either regular sets, including stationary points (equilibrium flow patterns), limit cycles (oscillating flow) and invariant tori (quasi-periodically oscillating flow), as well as irregular sets which are in fact strange attractors (chaotic flow). These sets of limit points are determined from a numerical bifurcation analysis of the spectral model, using adapted routines of the software package AUTO of DOEDEL [8].

A spectral model is assumed to give at least a qualitative description of the atmospheric circulation if trajectories irregularly visit different preferent regions in phase space. In this way the index cycle mentioned in the introduction is simulated. If this behaviour does not occur the truncation is apparently too severe and more modes should be included in the spectral expansions. Another possibility is to take account of the effect of the synoptic-scale transient eddies on the dynamics of planetary-specific scale flow by adding specific forcing terms to the spectral equations. However, this requires a thorough understanding of the interactions between different scales of motion, this being one of the major problems in modern dynamic meteorology. We will return to this point in the Sections 6 and 7.

5. A THREE- AND SIX-COMPONENT MODEL

The fact that the Galerkin projection technique, discussed in Section 3, can be applied to the partial differential equations describing the dynamics of large-scale atmospheric flow was first realized by SILBERMAN [26]. Later on a number of other spectral models have been developed, see the review in DE SWART [5]. It appears that already extremely low-order spectral models show qualitative features of the circulation. The simplest example is the three-component model of CHARNEY and DEVORE [2] in which only the $(0,1)$ zonal flow mode and the $(1,1)$ wave mode are retained. This implies that we assume ψ^*_{02} in (3.4) to be zero. The stationary points of this model can be computed analytically. There are either one or three of them depending on the model parameters. As a characteristic situation we will consider a channel of length 5000 km ($= 2\pi/k$) and width 4000 km centered at latitude $\phi_0=45°$ where $f_0=10^{-4}$ s^{-1} and $\beta_0=1.6\cdot10^{-11}$ m^{-1} s^{-1}. The vertical length scale is taken $H=10^4$ m, the time scale $\sigma^{-1}=10^5$ s (about one day), the mountain amplitude $h_0=10^3$ m and the dissipation time scale about ten days. This yields the parameter values $b=1.6$, $\beta=1.25$, $\gamma=1$ and $c=0.1$. In Figure 2 the x_1-component of the stationary points (where $x_1=\psi_{01}/b$) is presented as a function of the external forcing $x_1^*=\psi^*_{01}/b=U/U_0$, where U is a velocity scale for the forcing and $U_0=\sigma/k=7.8$ ms^{-1}.

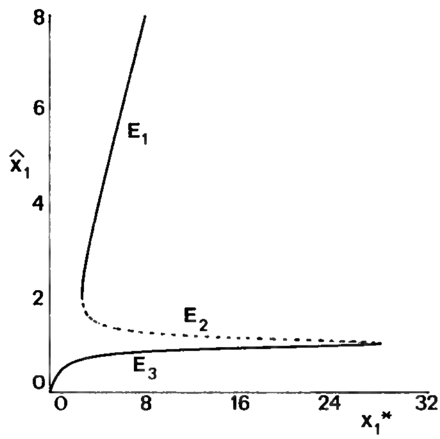

FIGURE 2. The x_1-component of the stationary points $\hat{x}$ of the three-component model for the parameter values discussed in the text. A solid line denotes that the solution is stable whereas a dashed line refers to an unstable solution.

In Figure 3 the streamfunction patterns associated with the equilibria E_1, E_2 and E_3 occurring for $x_1^*=4$ are shown. Note their strong resemblance to the circulation patterns shown in Figure 1. Based on this agreement CHARNEY and DEVORE [2] suggest that equilibria of spectral models indicate large-scale preferent states of the atmospheric circulation. The existence of multiple equilibria is a consequence of the presence of topography, forcing and dissipation.

However, no flow with a limited predictability and no vacillatory behaviour (i.e., an index cycle) is found: the nontransient solutions are always stationary. The reason for this behaviour is a lack of nonlinear interactions in the model.

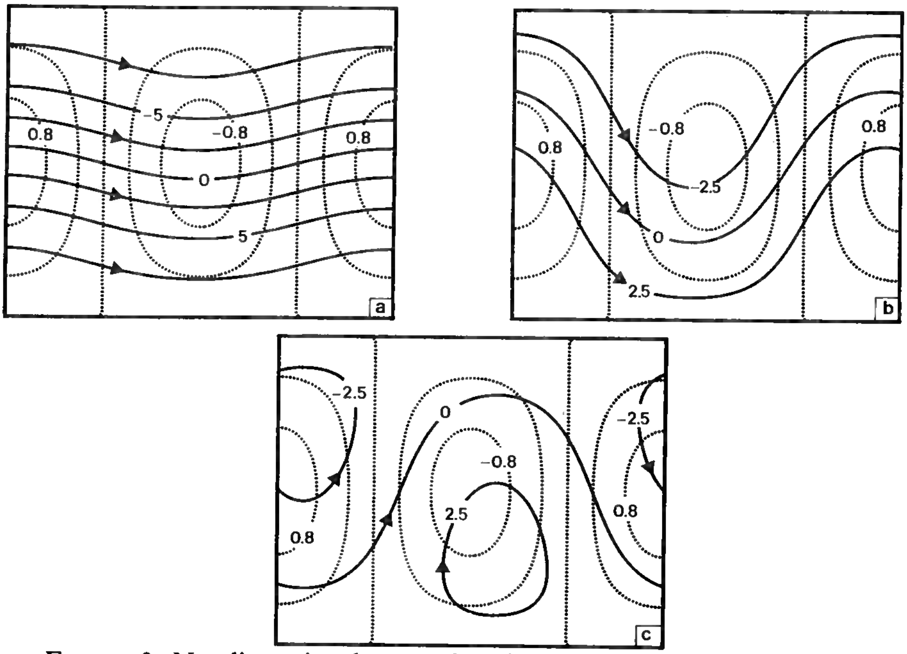

FIGURE 3. Nondimensional streamfunction patterns of the equilibria E_1 (a), E_2 (b) and E_3 (c) for the parameter values discussed in the text. The arrows indicate the flow direction which is along the streamlines ψ=constant. Here a difference $\Delta\psi=1$ corresponds to a zonal transport of $2.6 \cdot 10^7$ m^2 s^{-1}.The dashed lines represent contours of the topography (10^3 m).

Therefore, we extend the model by including also the (0,2) zonal flow mode and the (1,2) wave mode in the spectral expansions, resulting in a six-component model. In Figure 4 the x_1- and $x_4(=\psi_{02}/b)$-component of its stationary points are shown as a function of x_1^* in case $x_4^*(=\psi_{02}^*/b)=0$ and all other parameter values similar as before. Clearly, equilibria of the three-component model are also equilibria of the six-component model but stability properties can be different because of the increased number of degrees of freedom. Furthermore, the model contains a new type of nonlinear interactions involving a zonal flow component and two different wave modes. As a result additional equilibria are found. However, the nontransient behaviour can be more complicated. In DE SWART [5,6] it is shown that also stable periodic orbits exist, indicated by the presence of Hopf bifurcation points in Figure 4, as well as strange attractors. However, the latter have only a limited domain of attraction in phase space and the chaotic solutions remain in the low-index regime forever. Thus no simulation of an index cycle is obtained. This conclusion remains unchanged if x_4^* is given nonzero values. It is due to the presence of only one triad of nonlinear interactions in the model. Thus in order to obtain vacillating solutions either forcing terms must be added to the equations or more modes should be included in the spectral expansions. Both possibilities will be subsequently considered.

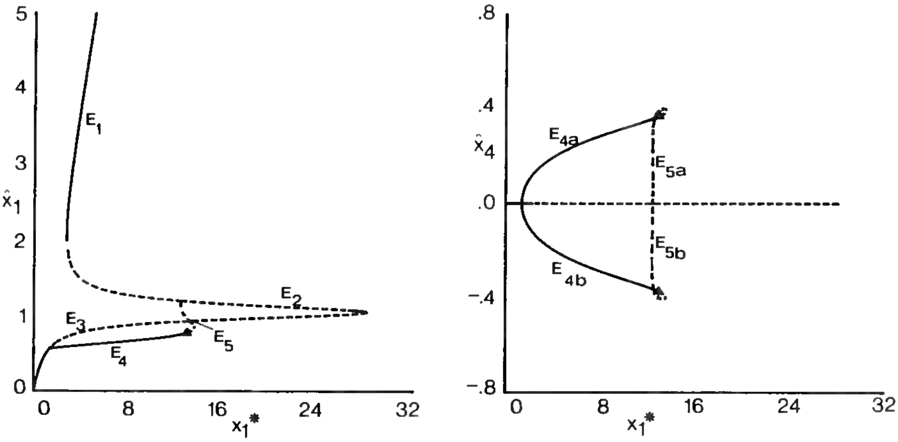

FIGURE 4. As Figure 2, but the x_1-component (left) and x_4-component (right) of the six-component model. A triangle denotes a Hopf bifurcation point.

6. EFFECT OF STOCHASTIC PERTURBATIONS ON LOW-ORDER SPECTRAL MODELS

A first attempt to model the effect of the unresolved modes on the three component model was carried out by EGGER [10]. He added stochastic forcing terms of Gaussian white noise with a fixed small intensity to the equations. The noise forces the system to visit alternately the attraction domains of the two stable equilibria, thus in this way an index cycle is simulated. A justification for choosing this type of forcing was given by EGGER and SCHILLING [11] who showed, using atmospheric data, that the forcing terms $F(t)$ in (4.1) can be modelled by coloured-noise processes. These are stationary and Gaussian Markov processes and contain white noise as a limit for the correlation time tending to zero. In DE SWART and GRASMAN [7] the effect of coloured-noise forcing on the three-component-model of Section 5 having three different stationary points is discussed. For simplicity we only consider the effect of white-noise forcing. Then Eq. (4.1) become

$$dx = f_u(x)dt + \epsilon dW \quad \text{in } \mathbb{R}^3, \tag{6.1}$$

where the three components of $W(t)$ are mutually independent Wiener processes and ϵ is the noise intensity which is assumed to be small ($\epsilon << 1$). Let the stable equilibria E_1 and E_3 have the attraction domains Ω_1 and Ω_3 with boundaries $\partial\Omega_1$ and $\partial\Omega_3$, respectively. We investigate the distribution of residence times $\tau(x)$ starting from a state $x \in \Omega_i$ $(i=1,3)$ of the system in these attraction domains. The expected value $<\tau(x)>=T(x)$ gives a measure of the persistence of a large-scale preferent state of the atmospheric circulation. In GORDINER [13] it is shown that $T(x)$ obeys

$$\frac{1}{2}\epsilon^2\nabla^2 T(x)+f_\mu(x)\cdot\nabla T(x) = -1 \text{ in } \Omega_i, \quad (6.2)$$

$$T=0 \text{ at } \partial\Omega_i \ (i=1 \text{ or } 3).$$

An asymptotic solution of this elliptic differential equation, valid for low-intensity noise ($\epsilon<<1$), is derived in MATKOWSKY ET AL. [20] by application of singular perturbation techniques. Outside a boundary layer near $\partial\Omega_i$ it reads

$$T_i \sim C_i e^{K_i/\epsilon^2}, \quad K_i = \lim_{x\to E_2} Q(x). \quad (6.3)$$

Here $Q(x)$ is the solution of the eikonal equation

$$\frac{1}{2}(\nabla Q(x))^2 + f_\mu(x)\cdot\nabla Q(x) = 0, \quad Q(E_i) = 0, \quad (6.4)$$

which can be solved by means of the ray method, see LUDWIG [19]. However, the computed residence times are of the order of months whereas from observations we expect life spans of weather regimes in the order of two weeks. We will discuss this indiscrepancy in Section 8. Furthermore, it is found that the most probable region of exit from the attraction domains is an ϵ-neighbourhood of the unstable equilibrium E_2. Here the system remains for a characteristic time

$$T_2 \sim \frac{1}{\lambda}\log(\frac{1}{\epsilon}), \quad (6.5)$$

where λ is the largest positive real part of the deterministic system linearized at E_2.

Once the stochastic dynamical system is in its statistical equilibrium it is characterized by the expected residence times in the different regimes. However, in this way no information is obtained about the time scale over which the effect of initial conditions is important. This can be investigated with a discrete-state Markov process model. For the randomly forced spectral models discussed here we can derive such a model with three states: a zonal state (1), a transitional state (2) and a meridional state (3). Let Q_{ij} denote the transition probability per unit of time from state i to j and let $p_i(t)$ denote the probability for the system to be in state i at time t. Then the $p_i(t)$ satisfy

$$\begin{aligned} \dot{p}_1 &= -(Q_{12}+Q_{21})p_1 - Q_{21}p_3 + Q_{21}, \\ \dot{p}_3 &= -Q_{23}p_1 - (Q_{32}+Q_{23})p_3 + Q_{23}, \\ p_2 &= 1-p_1-p_3, \end{aligned} \quad (6.6)$$

where

$$Q_{21} = Q_{23} = \frac{1}{2T_2}, \quad Q_{12} = \frac{1}{T_1}, \quad Q_{32} = \frac{1}{T_3}. \quad (6.7)$$

In Figure 5 the probability functions $p_i(t)$ are given for a process with $T_1=9$, $T_2=1$ and $T_3=31$ that starts in state 1, 2 and 3, respectively.

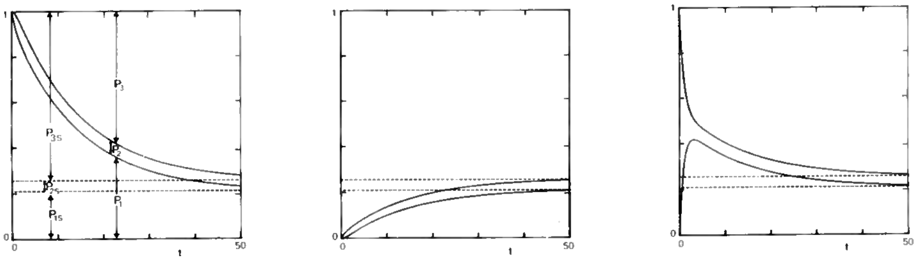

FIGURE 5. Evolution of the probability distribution of the Markov process starting in state 1 (left), 2 (middle) and 3 (right) respectively. The dotted lines represent the stationary distribution.

From these figures it is seen that once an initial state is given, the Markov model contains more information about the system than the stationary probability distribution for a period of about fifty days.

7. TEN COMPONENTS: DETERMINISTIC CHAOS AND VACILLATION

As discussed in Section 5, a second possibility for simulating an index cycle with spectral models of the quasi-geostrophic barotropic potential vorticity equation is to include more modes in the spectral expansions. LEGRAS and GHIL [16] have studied a 25-component model and found that solutions could visit different preferent regions in phase space. In DE SWART [6] a method is discussed to derive a 'minimum-order' spectral model which has, for fixed parameter values, multiple unstable regular solutions and a strange attractor. It is expected that trajectories starting from arbitrary initial conditions converge to this attractor. After that they must vacillate between different preferent regions in phase space which are close to the (weakly) unstable regular solutions. It is claimed that, by using a rectangular truncation of the eigenfunction expansions in wave number space, the minimum number of components is ten. The model describes the evolution of two zonal flow profiles (a (0,1) and (0,2) mode) and four waves (the (1,1), (1,2), (2,1) and (2,2) modes) in a barotropic atmosphere. Compared to the six-component model of Section 5 it contains a new type of nonlinear interaction involving three waves: the (1,1), (1,2) and (2,1) modes. In Figure 6 the x_1- and x_4-component of the stationary points of this model are shown for the same parameter values as discussed in Section 5. Due to the presence of the wave triad, isolated branches of equilibria occur. By letting x_4^* become nonzero all regular solutions may be turned unstable. For $x_4^*=-8$ the model represents a flow vacillating between three preferent regimes where the latter are actually unstable periodic solutions of the model, see Figure 7. The wave triad provides for interaction between clearly distinct scales of motion: a planetary scale and synoptic scale. This behaviour is similar to what is observed in the atmosphere.

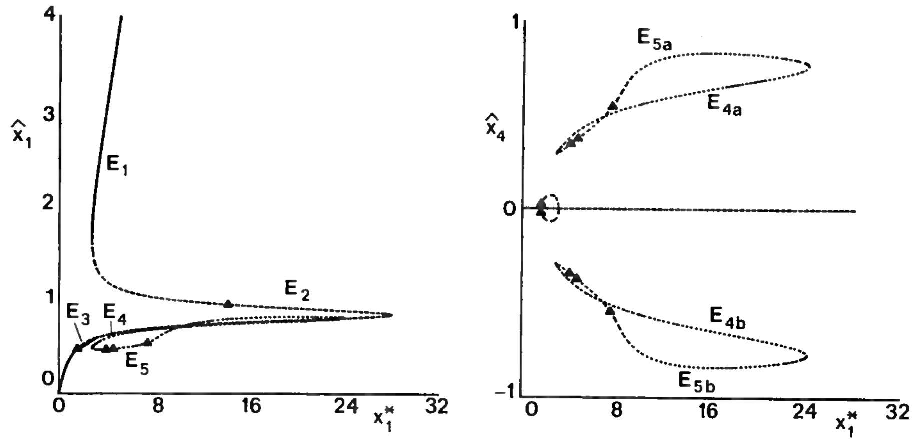

FIGURE 6. As Figure 4, but for the ten-component model.

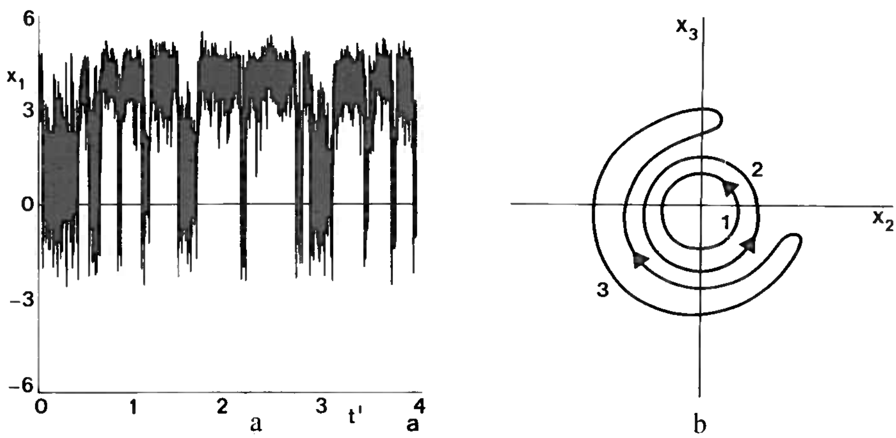

FIGURE 7a. The x_1-component of a chaotic solution of the ten-component model as a function of time. Here $t'=(t-1000)/500$ and the dimensional period is approximately six years.

b. Sketch of the unstable periodic orbits projected onto the x_2-x_3 plane. The preferent regions of the strange attractor are small tubes around these orbits.

By computing the spectrum of Lyapunov exponents, using the method of WOLF ET AL. [31] the existence of a global strange attractor is shown.

Lyapunov exponents measure the average divergence between nearby orbits in phase space whereas chaos is defined by at least one positive exponent. As discussed in FARMER ET AL. [12] from the spectrum of Lyapunov exponents we can estimate the fractal dimension of the attractor which also yields an upper bound to the number of degrees of freedom of the chaotic flow.

In practice initial conditions are never known with infinite precision. Thus small errors are introduced in the system which will grow during its evolution because of the chaotic dynamics. Consequently, the predictability of the flow is limited: a time scale of average prediction is given by the reciprocal of the sum of all positive Lyapunov exponents (SCHUSTER [25]). However, of more interest to meteorologists is the dependence of predictability on the state of the system (TENNEKES ET AL. [27]). In DE SWART [6] it is argued that the local eigenvalues at each point of an orbit may determine the time evolution of small errors on this orbit. In that case the eigenvectors corresponding to the eigenvalues with positive real part determine the geographical distribution of the error growth. However, this is on the condition that the time scale of error growth is small compared to the time scale on which the flow itself evolves. This method can be applied to spectral models showing long periods of quasi-stationary behaviour.

The impact of neglected short-scale modes on a planetary-scale model was studied by considering the chaotic ten-component model to represent the real atmosphere and the six-component model of Section 5 (for identical parameter values) to be a forecast model. For obtaining equivalence between solutions of the two systems forcing terms must be added to the equations of the forecast model. It appears that these forcing terms have an unpredictable nature and that they cannot be modelled by the simple stochastic processes used in Section 6. We will discuss these results in the next section.

8. CONCLUDING REMARKS

In this final section we briefly discuss the relevance of our investigations to a better understanding of the atmospheric circulation. It was remarked in the introduction that an accurate modelling of the feedback between quasi-stationary planetary-scale motion and transient synoptic-scale eddies is important for the development of long-range weather forecast models. Here we have argued that this problem may be studied by considering simplified models which still represent the chaotic properties and vacillatory behaviour of the atmosphere. Next we investigated whether they provide clues on how to analyse more complicated models as well as real data.

As discussed in Section 5, already extremely low-order spectral models show qualitative features of the atmospheric circulation. They possess multiple equilibria for a range of parameter values and the corresponding flow patterns resemble large-scale preferent states of the atmospheric circulation. However, we remark that the existence of weather regimes has never been convincingly demonstrated by a systematic data analysis; only recently some indications have been found (BENZI ET AL. [1]).

It has been found that the three- and six-component models cannot simulate a flow vacillating between different weather regimes. In order to meet this imperfection, in Section 6 stochastic forcing terms were added to the spectral equations. They are assumed to account for the effect of neglected modes and physical processes not incorporated in the model. A justification for choosing white noise and coloured noise parametrizations is found from the data study of EGGER and SCHILLING [11]. The noise forces the system to visit alternately attraction domains of the stable equilibria. During a transition the system remains a characteristic time near an unstable equilibrium. This suggests that stable and unstable regular solutions of a spectral model may have some relevance for the dynamics of the atmospheric circulation. A method was discussed for computing expected residence times near the equilibria of the unperturbed system. Comparing the results with observational data it appears that the computed life spans of the weather regimes are a factor of 10 larger than those in the atmosphere.

A systematic way for investigating the effect of neglected modes on a truncated spectral model has been discussed in Section 7. Here a ten-component model is considered which is a 'minimum-order' model representing a finitely predictable flow having two distinct scales of motion (a planetary and synoptic scale) and vacillating between different preferent regimes. We assumed this model to represent the real atmosphere and considered a six-component subsystem as a forecast model. To the subsystem forcing terms were added such that its solutions are equivalent to those of the full model projected onto the modes which also belong to the subsystem. It was found that these forcing terms have an unpredictable nature and that they cannot be modelled by coloured-noise processes. This result is in agreement with that of LINDENBERG and WEST [17], who analysed explicit expressions for the forcing terms representing the effect of the neglected modes on truncated spectral models of the barotropic vorticity equation. It does not contradict the result of EGGER and SCHILLING [11] since the latter authors also include the effect of neglected physical processes in their definition of the forcing terms.

We remark that effects of topography are over-estimated in barotropic models since they act directly on the entire fluid column. Baroclinic multi-level models of the quasi-geostrophic potential vorticity equation give better results at this point. Again multiple equilibria are found (CHARNEY and STRAUS [3]) and due to the presence of baroclinic instability mechanisms vacillatory behaviour is even more easily produced. The two-level twenty-component model of REINHOLD and PIERREHUMBERT [24] is probably the simplest model containing all basic physical mechanisms: topographic, barotropic and baroclinic instability as well as the occurrence of wave triad interactions.

The problems with low-order spectral models in general is that unrealistically large external forcing values (corresponding to an equator-pole temperature difference of more than 150° C) are required in order to produce vacillatory behaviour. Moreover, the characteristic lives of the regimes in the models are much larger than those obtained from atmospheric data. These imperfections are probably due to the severe truncation in both the horizontal and vertical direction. A better description of the atmospheric circulation is expected from multi-level high resolution models. Since their structure is extremely complicated they are difficult to analyse. Alternatively, we can study lower-dimensional spectral models which include an appropriate parametrization of the synoptic forcing terms. This problem remains to be investigated in more detail.

ACKNOWLEDGEMENTS

I would like to thank Dr. J. Grasman and Dr. J.D. Opsteegh (both at the State University of Utrecht) for their various and valuable contributions to the research carried out in this project.

REFERENCES

1. R. BENZI, P. MALGUZZI, A. SPERANZA, A. SUTERA (1986). The statistical properties of the general atmospheric circulation: observational evidence on a minimal theory of bimodality. *Q.J.R. Meteorol. Soc. 112,* 661-674.
2. J.G. CHARNEY, J.G. DEVORE (1979). Multiple flow equilibria in the atmosphere and blocking. *J. Atmos. Sci. 36,* 1205-1216.
3. J.G. CHARNEY, D.M. STRAUS (1980). Form-drag instability, multiple equilibria and propagating planetary waves in baroclinic, orographically forced planetary wave systems. *J. Atmos. Sci. 37,* 1157-1176.
4. P. CONSTANTIN, C. FOIAS, O.P. MANLEY, R. TEMAM (1985). Determining modes and fractal dimensions of turbulent flows. *J. Fluid Mech. 150,* 427-440.
5. H.E. DE SWART (1988). Low-order spectral models of the atmospheric circulation: a survey. *Acta Appl. Math. 11,* 49-96.
6. H.E. DE SWART (1988). On the vacillation behaviour and predictability properties of low-order atmospheric spectral models. Thesis, to appear.
7. H.E. DE SWART, J. GRASMAN (1987). Effect of stochastic perturbations on a low-order spectral model of the atmospheric circulation. *Tellus 39A,* 10-24.
8. E.J. DOEDEL (1986). *AUTO 86 User Manual, Software for Continuation and Bifurcation Problems in Ordinary Differential Equations,* Concordia University, Montreal.
9. R.M. DOLE (1986). Persistent anomalies of the extratropical Northern Hemisphere wintertime circulation: structure. *Mon. Wea. Rev. 114,* 178-207.
10. J. EGGER (1981). Stochastically driven large-scale circulations with multiple equilibria. *J. Atmos. Sci. 38,* 2608-2618.
11. J. EGGER, H.D. SCHILLING (1983). On the theory of the long-term variability of the atmosphere. *J. Atmos. Sci. 40,* 1073-1085.
12. D.J. FARMER, E. OTT, J.A. YORKE (1983). The dimension of chaotic attractors. *Physica 7D,* 153-180.
13. C.W. GORDINER (1983). *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences,* Springer Verlag, Berlin.
14. J. GUCKENHEIMER, P. HOLMES (1983). *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields,* Springer Verlag, New York.

15. M. Jarraud, A.P.M. Baede (1985). The use of spectral techniques in numerical weather prediction. *Lect. Appl. Math. 22,* 1-41.
16. B. Legras, M. Ghil (1985). Persistent anomalies, blocking and variations in atmospheric predictability. *J. Atmos. Sci. 42,* 433-471.
17. K. Lindenberg, B.J. West (1984). Fluctuations and dissipation in a barotropic flow field. *J. Atmos. Sci. 41,* 3021-3031.
18. E.N. Lorenz (1969). The predictability of a flow which possesses many scales of motion. *Tellus 21,* 289-307.
19. D. Ludwig (1975). Persistence of dynamical systems under random perturbations. *SIAM Rev. 17,* 605-640.
20. B.J. Matkowsky, Z. Schuss, C. Tier (1983). Diffusion across characteristic boundaries with critical points. *SIAM J. Appl. Math. 43,* 673-695.
21. J.D. Opsteegh (1988). De voorspelbaarheid van het weer. *Natuur en Techniek.* To appear.
22. J. Pedlosky (1987). *Geophysical Fluid Dynamics,* 2nd edition, Springer Verlag, New York.
23. B.B. Reinhold (1987). Weather regimes: the challenge in extended-range forecasting. *Science 235,* 437-441.
24. B.B. Reinhold R.T. Pierrehumbert (1982). Dynamics of weather regimes: quasi-stationary waves and blocking. *Mon. Wea. Rev. 110,* 1105-1145.
25. H.G. Schuster (1984). *Deterministic Chaos, an Introduction,* Physik Verlag, G.M.B.H., Waldheim.
26. I. Silberman (1954). Planetary waves in the atmosphere. *J. Meteorol. 11,* 27-34.
27. H. Tennekes, A.P.M. Baede, J.D. Opsteegh (1986). Forecasting forecast skill. Proc. ECMWF Workshop *On predictability in the medium and extended range,* 17-19 March 1986, ECMWF, Reading, 277-302.
28. J.M.T. Thompson, H.B. Stewart (1986). *Nonlinear Dynamics and Chaos: Geometrical Methods for Engineers and Scientists,* Wiley, Chichester.
29. W. Van Dijk, F.H. Schmidt, C.J.E. Schuurmans (1974). Beschrijving en toepassingsmogelijkheden van gemiddelde topografieën van het 500 mb-vlak in afhankelijkheid van circulatietypen. KNMI, de Bilt, Wetenschappelijk Rapport WR 74-3.
30. R.G. Voigt, D. Gottlieb, M. Yousoff-Hussaini (eds.) (1984). *Spectral Methods for Partial Differential Equations,* SIAM, Philadelphia.
31. A. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano (1985). Determining Lyapunov exponents from a time series. *Physica 6D,* 285-317.